Integration 2

1. Find:

(i) $\int\left(1-\sqrt{x}\right)dx$ **(ii)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{3}\sin x\,dx$ **(iii)** $\int_0^{\frac{\pi}{4}}-\cos 2x\,dx$ **(iv)** $\int 6e^{3x+5}dx$

(v) $\int x\sqrt{4-2x^2}dx$ **(vi)** $\int x\cos x\,dx$

(i) $\int\left(1-\sqrt{x}\right)dx=\boldsymbol{x-\frac{2}{3}x^{\frac{3}{2}}+c}$

(ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{3}\sin x\,dx=\left[-\sqrt{3}\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=-\sqrt{3}\cos\frac{\pi}{2}+\sqrt{3}\cos\frac{\pi}{6}=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)=\boldsymbol{\frac{3}{2}}$

(iii) $\int_0^{\frac{\pi}{4}}-\cos 2x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}-\cos 2x\,d(2x)=\left[-\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{4}}=\left[-\frac{\sin\left(2\times\frac{\pi}{4}\right)}{2}\right]-\left[-\frac{\sin(0)}{2}\right]=\boldsymbol{-\frac{1}{2}}$

(iv) $\int 6e^{3x+5}dx=\frac{6}{3}\int e^{3x+5}d(3x+5)=\boldsymbol{2e^{3x+5}}+\boldsymbol{c}$

(v) $\int x\sqrt{4-2x^2}dx=-\frac{1}{4}\int\sqrt{4-2x^2}d(4-2x^2)=-\frac{1}{4}\times\frac{2}{3}(4-2x^2)^{\frac{3}{2}}+c=\boldsymbol{-\frac{1}{6}(4-2x^2)^{\frac{3}{2}}+c}$

(vi) $\int x\cos x\,dx=\int x d(\sin x)=x\sin x-\int\sin x\,dx=\boldsymbol{x\sin x+\cos x+c}$

2. Evaluate $\int sin^2\,x\,cos^2\,x\,dx$.

$\int sin^2\,x\,cos^2\,x\,dx=\frac{1}{4}\int(2\sin x\cos x)^2\,dx=\frac{1}{4}\int\sin^2 2x\,dx=\frac{1}{8}\int(1-\cos 4x)\,dx$

$=\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+c$

3. Evaluate $\int x^2\,cos^{-1}x\,dx$.

$\int x^2\,cos^{-1}x\,dx=\frac{1}{3}\int cos^{-1}x\;\;d(x^3)=\frac{1}{3}[x^3cos^{-1}x-\int x^3\;\;d(cos^{-1}x)]$, integration by parts

$=\frac{1}{3}\left[x^3cos^{-1}x-\int x^3\left(-\frac{dx}{\sqrt{1-x^2}}\right)\right]=\frac{1}{3}\left[x^3cos^{-1}x+\int\frac{x^3}{\sqrt{1-x^2}}dx\right]$

$=\frac{1}{3}\left[x^3cos^{-1}x-\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}d(1-x^2)\right]$

$=\frac{1}{3}\left[x^3cos^{-1}x-\frac{1}{2}\int\frac{1-(1-x^2)}{\sqrt{1-x^2}}d(1-x^2)\right]$

$=\frac{1}{3}\left[x^3cos^{-1}x-\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}d(1-x^2)+\frac{1}{2}\int(1-x^2)^{\frac{1}{2}}d(1-x^2)\right]$

$=\frac{1}{3}\left[x^3cos^{-1}x-\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}+\frac{1}{2}\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$

$$= \frac{1}{3}\left[x^3 cos^{-1}x - (1-x^2)^{\frac{1}{2}} + \frac{(1-x^2)^{\frac{3}{2}}}{3}\right] + c$$

4. Find by integration by parts $\int x^3\, e^{-x^2} dx$.

$$\int x^3\, e^{-x^2} dx = -\frac{1}{2}\int x^2\, d\left(e^{-x^2}\right) = -\frac{1}{2}\left[x^2 e^{-x^2} - \int e^{-x^2} d(x^2)\right] = -\frac{1}{2}\left[x^2 e^{-x^2} - \int 2xe^{-x^2} dx\right]$$

$$= -\frac{1}{2}\left[x^2 e^{-x^2} + e^{-x^2}\right] + c = -\frac{1}{2} e^{-x^2}[x^2+1] + c$$

5. Evaluate $\int_0^{\pi} \frac{d\theta}{a^2 - 2ab\cos\theta + b^2}$

Let $t = \tan\frac{\theta}{2}$, $\quad dt = \frac{1}{2}\sec^2\frac{\theta}{2} d\theta \Rightarrow d\theta = \frac{2dt}{\sec^2\frac{\theta}{2}} = \frac{2dt}{1+t^2}$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}} = \frac{1 - \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} + 1} = \frac{1-t^2}{1+t^2}$$

When $\theta = \pi,\ t = \infty$. When $\theta = 0,\ t = 0$

$$\int_0^{\pi} \frac{d\theta}{a^2 - 2ab\cos\theta + b^2} = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{a^2 - 2ab\left(\frac{1-t^2}{1+t^2}\right) + b^2} = \int_0^{\infty} \frac{2dt}{(a^2+b^2)(1+t^2) - 2ab(1-t^2)} = \int_0^{\infty} \frac{2dt}{(a-b)^2 + (a+b)^2 t^2}$$

$$= \frac{2}{(a+b)^2}\int_0^{\infty} \frac{dt}{\left(\frac{a-b}{a+b}\right)^2 + t^2} = \frac{2}{(a+b)^2}\left[\frac{a+b}{a-b} tan^{-1}\left(\frac{a+b}{a-b}\right) t\right]_0^{\infty} = \frac{1}{a^2 - b^2}\left(\frac{\pi}{2}\right), a > b$$

6. Evaluate $\int \cos 2x \cot 2x\, dx$

$$\int \cos 2x \cot 2x\, dx = \int \frac{\cos^2 2x}{\sin 2x} dx = \int \frac{1-\sin^2 2x}{\sin 2x} = \int \left[\frac{1}{\sin 2x} - \sin 2x\right] dx = \int \frac{dx}{2\sin x\cos x} - \int \sin 2x dx$$

$$= \frac{1}{2}\int \frac{\sec^2 x}{\tan x} dx - \int \sin 2x dx = \frac{1}{2}\int \frac{d(\tan x)}{\tan x} - \int \sin 2x dx = \frac{1}{2}\ln|\tan x| + \frac{1}{2}\cos 2x + c$$

7. Evaluate $\int \frac{\ln x}{x^5} dx$

$I = \int \frac{\ln x}{x^5} dx = -\frac{1}{4}\int \ln x\, d\left(\frac{1}{x^4}\right) = -\frac{1}{4}\left[\frac{1}{x^4}\ln x - \int \frac{1}{x^4} d(\ln x)\right]$, integration by parts

$$= -\frac{1}{4}\left[\frac{1}{x^4}\ln x - \int \frac{1}{x^4}\left(\frac{dx}{x}\right)\right] = -\frac{1}{4x^4}\ln x + \frac{1}{4}\int \frac{1}{x^5} dx = -\frac{1}{4x^4}\ln x - \frac{1}{16x^4} + c$$

(You may put $u = \ln x, dv = \frac{dx}{x^5}$ in the integration by parts.)

8. Evaluate $\int (3t + t^2)\sin(2t)\,dt$.

$$\int (3t + t^2)\sin(2t)\,dt = -\frac{1}{2}\int (3t + t^2)\,d[cos(2t)] = -\frac{1}{2}[(3t + t^2)cos(2t) - \int cos(2t)\,d(3t + t^2)]$$

$$= -\frac{1}{2}(3t + t^2)cos(2t) + \frac{1}{2}\int (2t + 3)cos(2t)\,dt = -\frac{1}{2}(3t + t^2)cos(2t) + \frac{1}{4}\int (2t + 3)\,d[\sin(2t)]$$

$$= -\frac{1}{2}(3t + t^2)cos(2t) + \frac{1}{4}[(2t + 3)\sin(2t) - \int \sin(2t) d\,(2t + 3)]$$

$$= -\frac{1}{2}(3t + t^2)cos(2t) + \frac{1}{4}[(2t + 3)\sin(2t) - 2\int \sin(2t)\,dt]$$

$$= -\frac{1}{2}(3t + t^2)cos(2t) + \frac{1}{4}[(2t + 3)\sin(2t) + cos(2t)] + c$$

$$= \frac{1}{4}[(2t + 3)\sin(2t) - (2t^2 + 6t - 1)cos(2t)] + c$$

9. Evaluate $\int x^3 \ln x\,dx$.

$$\int x^3 \ln x\,dx = \frac{1}{4}\int \ln x\,d(x^4) = \frac{1}{4}[x^4 \ln x - \int x^4\,d(\ln x)] = \frac{1}{4}\left[x^4 \ln x - \int x^4 \frac{dx}{x}\right]$$

$$= \frac{1}{4}[x^4 \ln x - \int x^3\,dx] = \frac{1}{4}\left[x^4 \ln x - \frac{x^4}{4}\right] + c = \frac{1}{16}[4x^4 \ln x - x^4] + c$$

10. Evaluate $\int \mathrm{t}^2 \ln(1 + \mathrm{t}^3)\,dt$.

$$\int \mathrm{t}^2 \ln(1 + \mathrm{t}^3)\,dt = \frac{1}{3}\int \ln(1 + \mathrm{t}^3)\,d(1 + \mathrm{t}^3)$$

$$= \frac{1}{3}[(1 + \mathrm{t}^3)\ln(1 + \mathrm{t}^3) - \int (1 + \mathrm{t}^3)\,d\ln(1 + \mathrm{t}^3)] \qquad \text{(integration by parts)}$$

$$= \frac{1}{3}\left[(1 + \mathrm{t}^3)\ln(1 + \mathrm{t}^3) - \int (1 + \mathrm{t}^3)\frac{3\mathrm{t}^2}{1+\mathrm{t}^3}\mathrm{dt}\right] = \frac{1}{3}[(1 + \mathrm{t}^3)\ln(1 + \mathrm{t}^3) - \int 3\mathrm{t}^2 dt]$$

$$= \frac{1}{3}[(1 + \mathrm{t}^3)\ln(1 + \mathrm{t}^3) - \mathrm{t}^3] + c$$

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